Available online at www.sciencedirect.com
ScienceDirect
Computers and Mathematics with Applications 55 (2008) 535–549
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Some families of linear operators associated with certain subclasses of multivalent functions

M.K. Aouf[a], H. Silverman[b], H.M. Srivastava[c,*]

[a] *Department of Mathematics, Faculty of Science, University of Mansoura, Mansoura 35516, Egypt*
[b] *Department of Mathematics, College of Charleston, Charleston, SC 29424-0001, USA*
[c] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada*

Abstract

Making use of a certain linear operator, which is defined here by means of the Hadamard product (or convolution), we introduce two novel subclasses

$$\mathcal{P}_{a,c}(A, B; p, \lambda) \quad \text{and} \quad \mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$$

of the class $\mathcal{A}(p)$ of normalized p-valent analytic functions in the open unit disk. The main objective of the present paper is to investigate the various important properties and characteristics of each of these subclasses. Furthermore, several properties involving neighborhoods of functions in these subclasses are investigated. We also derive many results for the modified Hadamard products of functions belonging to the class $\mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$. Finally, some applications of fractional calculus operators are considered.

Keywords: Linear operators; Analytic functions; Multivalent functions; Neighborhoods of analytic functions; Fractional calculus operators; Jack's lemma; Cauchy–Schwarz inequality

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions normalized by

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

* Corresponding author.
E-mail addresses: mkaouf127@yahoo.com (M.K. Aouf), silvermanh@cofc.edu (H. Silverman), harimsri@math.uvic.ca (H.M. Srivastava).

0898-1221/$ - see front matter
doi:10.1016/j.camwa.2007.06.001

If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written symbolically as follows:

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$ in $\mathbb{U}$ such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

For functions $f(z) \in \mathcal{A}(p)$, given by (1.1), and $g(z) \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \quad (p \in \mathbb{N}), \tag{1.2}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) := z^p + \sum_{k=1}^{\infty} a_{k+p}\, b_{k+p} z^{k+p} =: (g * f)(z) \quad (z \in \mathbb{U}). \tag{1.3}$$

Next, in terms of the Pochhammer symbol $(\nu)_n$ given by

$$(\nu)_n := \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1 & (n=0) \\ \nu(\nu+1)\cdots(\nu+n-1) & (n \in \mathbb{N}), \end{cases}$$

we define the function $\varphi_p(a, c; z)$ by

$$\varphi_p(a, c; z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p} \quad (a \in \mathbb{R};\, c \in \mathbb{R} \setminus \mathbb{Z}_0^-;\, \mathbb{Z}_0^- := \{0, -1, -2, \ldots\};\, z \in \mathbb{U}). \tag{1.4}$$

Corresponding to the function $\varphi_p(a, c; z)$, we consider a linear operator $L_p(a, c)$ which is defined by means of the following Hadamard product (or convolution):

$$L_p(a, c) f(z) = \varphi_p(a, c; z) * f(z) \quad (f \in \mathcal{A}(p)). \tag{1.5}$$

The operator $L_p(a, c)$ was introduced by Saitoh [1] and studied recently by (for example) Srivastava and Patel [2]. It is easily verified from (1.4) and (1.5) that

$$z(L_p(a, c) f(z))' = a L_p(a+1, c) f(z) - (a-p) L_p(a, c) f(z). \tag{1.6}$$

Moreover, for $f(z) \in \mathcal{A}(p)$, we observe that

$$L_p(a, a) f(z) = f(z), \qquad L_p(p+1, p) f(z) = \frac{z f'(z)}{p}$$

and

$$L_p(n+p, 1) f(z) = \mathcal{D}^{n+p-1} f(z) \quad (n > -p),$$

where $\mathcal{D}^{n+p-1} f(z)$ denotes the Ruscheweyh derivative of a function $f(z) \in \mathcal{A}(p)$ of order $n+p-1$ (see [3,4]).

For $p \in \mathbb{N}$, $a > 0$ and $c > 0$, and for the parameters λ, A and B such that

$$-1 \leqq B < A \leqq 1 \quad \text{and} \quad 0 \leqq \lambda < p, \tag{1.7}$$

we say that a function $f(z) \in \mathcal{A}(p)$ is in the class $\mathcal{P}_{a,c}(A, B; \lambda, p)$ if it satisfies the following subordination condition:

$$\frac{1}{p-\lambda}\left(\frac{(L_p(a, c) f(z))'}{z^{p-1}} - \lambda\right) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U}) \tag{1.8}$$

or, equivalently, if the following inequality holds true:

$$\left| \frac{\frac{(L_p(a,c) f(z))'}{z^{p-1}} - p}{B \frac{(L_p(a,c) f(z))'}{z^{p-1}} - [pB + (A-B)(p-\lambda)]} \right| < 1 \quad (z \in U). \tag{1.9}$$

By specializing the parameters a, c, A, B, p and λ involved in the class $\mathcal{P}_{a,c}(A, B; p, \lambda)$, we obtain the following subclasses which were studied in many earlier works:

(i) $\mathcal{P}_{a,a}(A, B; p, \lambda) = \mathcal{S}_p(A, B, \lambda)$ (Aouf [5]);
(ii) $\mathcal{P}_{a,a}(-1, 1; p, \lambda) = \mathcal{S}_p(\lambda)$ (Owa [6]);
(iii) $\mathcal{P}_{a,a}(A, B; p, 0) = \mathcal{S}_p(A, B)$ (Chen [7]);
(iv) $\mathcal{P}_{n+p,1}(-1, 1; p, \lambda) = \mathcal{T}_{n+p-1}(\lambda)$ $(n > -p;\ p \in \mathbb{N};\ 0 \leqq \lambda < p)$ (Goel and Sohi [8]).
(v) $\mathcal{P}_{n+p,1}(-A, -B; p, 0) = \mathcal{V}_{n,p}(A, B)$ $(-1 \leqq B < A \leqq 1;\ n > -p;\ p \in \mathbb{N})$ (Kumar and Shukla [4]);
(vi) $\mathcal{P}_{n+p,1}(-A, -B; p, \lambda) = \mathcal{V}_{n,p}(A, B, \lambda)$ $(-1 \leqq B < A \leqq 1;\ n > -p;\ p \in \mathbb{N};\ 0 \leqq \lambda < p)$ (Aouf [9]).
(vii) $\mathcal{P}_{a,a}(-A, -B; 1, 0) = \mathcal{R}(A, B)$ $(-1 \leqq B < A \leqq 1)$ (Mehrok [10]).

Furthermore, we say that a function $f(z) \in \mathcal{P}_{a,c}(A, B; p, \lambda)$ is in the subclass $\mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$ if $f(z)$ is of the following form:

$$f(z) = z^p - \sum_{k=1}^{\infty} |a_{k+p}| z^{k+p} \quad (p \in \mathbb{N}). \tag{1.10}$$

Thus, by specializing the parameters a, c, A, B, p and λ, we obtain the following familiar subclasses of analytic functions in $\mathbb{U}$ with *negative* coefficients:

(i) $\mathcal{P}^{+}_{a,a}(A, B; p, \lambda) = \mathcal{P}^{*}(p, A, B, \lambda)$ (Aouf [11]);
(ii) $\mathcal{P}^{+}_{a,a}(-\beta, \beta; p, \lambda) = \mathcal{P}^{*}_{p}(\lambda, \beta)$ $(0 \leqq \lambda < p;\ 0 < \beta \leqq 1)$ (Aouf [12]);
(iii) $\mathcal{P}^{+}_{a,a}(A, B; p, 0) = \mathcal{P}^{*}(p, A, B)$ (Shukla and Dashrath [13]);
(iv) $\mathcal{P}^{+}_{a,a}(-1, 1; p, \lambda) = \mathcal{F}_p(1, \lambda)$ $(0 \leqq \lambda < p;\ \ p \in \mathbb{N})$ (Lee et al. [14]);
(v) $\mathcal{P}^{+}_{a,a}(-\beta, \beta; 1, \lambda) = \mathcal{P}^{*}(\lambda, \beta)$ $(0 \leqq \lambda < 1;\ \ 0 < \beta \leqq 1)$ (Gupta and Jain [15]);
(vi) $\mathcal{P}^{+}_{n+p,1}(-1, 1; p, \lambda) = \mathcal{Q}_{n+p-1}(\lambda)$ $(n > -p;\ p \in \mathbb{N};\ 0 \leqq \lambda < p)$ (Aouf and Darwish [16]);
(vii) $\mathcal{P}^{+}_{n+1,1}(-1, 1; 1, \lambda) = \mathcal{Q}_n(\lambda)$ $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ 0 \leqq \lambda < 1)$ (Uralegaddi and Sarangi [17]).

In our present paper, we shall make use of the familiar *integral operator* $J_{\delta,p}$ defined by (*cf.* [18–20]; see also [21])

$$(J_{\delta,p} f)(z) := \frac{\delta + p}{z^p} \int_0^z t^{\delta-1} f(t)\mathrm{d}t \quad (f \in \mathcal{A}(p);\ \delta > -p;\ p \in \mathbb{N}) \tag{1.11}$$

as well as the *fractional calculus operator* D^{μ}_{z} for which it is well known that (see, for details, [22] and [23]; see also Section 5 below)

$$D^{\mu}_{z}\{z^{\rho}\} = \frac{\Gamma(\rho + 1)}{\Gamma(\rho - \mu + 1)} z^{\rho-\mu} \quad (\rho > -1;\ \mu \in \mathbb{R}) \tag{1.12}$$

in terms of Gamma functions.

Remark 1. Throughout our present investigation, we tacitly assume that the parametric constraints listed in (1.7) are satisfied.

2. Inclusion properties of the function class $\mathcal{P}_{a,c}(A, B; \lambda, p)$

For proving our first inclusion result, we shall make use of the following lemma.

Lemma 1 (*Jack's Lemma [24]*). *Let the nonconstant function $w(z)$ be analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then*

$$z_0 w'(z_0) = \gamma w(z_0), \tag{2.1}$$

where γ is a real number and $\gamma \geqq 1$.

Theorem 1. *If $a > 0$, then*

$$\mathcal{P}_{a+1,c}(A, B; p, \lambda) \subset \mathcal{P}_{a,c}(A, B; p, \lambda).$$

Proof. If $f(z)$ is in the class $\mathcal{P}_{a+1,c}(A, B; p, \lambda)$, then we find from (1.8) that

$$\frac{z(L_p(a+1,c)f(z))'}{z^p} = \frac{p+[pB+(A-B)(p-\lambda)]w_1(z)}{1+Bw_1(z)}, \tag{2.2}$$

where $w_1(z)$ is a Schwarz function. To prove that $f(z)$ is in the class $\mathcal{P}_{a,c}(A, B; p, \lambda)$, we write

$$\frac{z(L_p(a,c)f(z))'}{z^p} = \frac{p+[pB+(A-B)(p-\lambda)]w(z)}{1+Bw(z)}. \tag{2.3}$$

It now suffices to show that $|w(z)| < 1$. Indeed, by using (1.6) and (2.3), we have

$$\frac{(L_p(a+1,c)f(z))'}{z^{p-1}} = \frac{p+[pB+(A-B)(p-\lambda)]\,w(z)}{1+Bw(z)} + \frac{(A-B)(p-\lambda)zw'(z)}{a[1+Bw(z)]^2}. \tag{2.4}$$

We claim that

$$|w(z)| < 1 \quad (z \in \mathbb{U}).$$

Otherwise, there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leqq |z_0|} |w(z)| = |w(z_0)| = 1.$$

Applying Jack's Lemma, we have

$$z_0 w'(z_0) = \gamma w(z_0) \quad (\gamma \geqq 1).$$

Now, upon setting

$$w(z_0) = \mathrm{e}^{\mathrm{i}\theta} \quad (0 \leqq \theta \leqq 2\pi),$$

if we put $z = z_0$ in (2.4), we get

$$\left|\frac{\frac{(L_p(a+1,c)f(z_0))'}{z_0^{p-1}} - p}{B\frac{(L_p(a+1,c)f(z_0))'}{z_0^{p-1}} - [pB+(A-B)(p-\lambda)]}\right|^2 - 1 = \frac{|(a+\gamma)+aB\mathrm{e}^{\mathrm{i}\theta}|^2 - |a+B(a-\gamma)\mathrm{e}^{\mathrm{i}\theta}|^2}{|a+B(a-\gamma)\mathrm{e}^{\mathrm{i}\theta}|^2}$$

$$= \frac{\gamma^2(1-B^2)+2a\gamma(1+B^2+2B\cos\theta)}{|a+B(a-\gamma)\mathrm{e}^{\mathrm{i}\theta}|^2} \geqq 0,$$

which, in view of (1.9), contradicts our hypothesis that

$$f(z) \in \mathcal{P}_{a+1,c}(A, B; p, \lambda).$$

Thus we must have

$$|w(z)| < 1 \quad (z \in \mathbb{U}).$$

So, by applying (2.3), we conclude that

$$f(z) \in \mathcal{P}_{a,c}(A, B; p, \lambda).$$

This completes the proof of Theorem 1. □

Theorem 2. *Let ν be a complex number with $\Re(\nu) > 0$. If*

$$f(z) \in \mathcal{P}_{a,c}(A, B; p, \lambda),$$

then the function $F(z)$ given by

$$F(z) = \frac{\nu+p}{z^\nu}\int_0^z t^{\nu-1}f(t)\mathrm{d}t \tag{2.5}$$

is also in the class $\mathcal{P}_{a,c}(A, B; p, \lambda)$.

Proof. From (2.5), we have

$$(\nu + p)L_p(a,c)f(z) = \nu L_p(a,c)F(z) + z(L_p(a,c)F(z))'. \tag{2.6}$$

Let

$$\frac{(L_p(a,c)F(z))'}{z^{p-1}} = \frac{p + [pB + (A-B)(p-\lambda)]w(z)}{1 + Bw(z)}, \tag{2.7}$$

where $w(z)$ is either analytic or meromorphic in $\mathbb{U}$ with $w(0) = 0$. Then, by differentiating (2.7) and using (2.6), we obtain

$$\frac{(L_p(a,c)f(z))'}{z^{p-1}} = \frac{p + [pB + (A-B)(p-\lambda)]w(z)}{1 + Bw(z)} + \frac{(A-B)(p-\lambda)zw'(z)}{(\nu + p)[1 + Bw(z)]^2}. \tag{2.8}$$

The remaining part of the proof of Theorem 2 is much akin to that of Theorem 1, and so it is being omitted here. □

Theorem 3. *Let $f(z) \in \mathcal{A}(p)$. Then $f(z) \in \mathcal{P}_{a,c}(A, B; p, \lambda)$ if and only if the function $F(z)$ given by*

$$F(z) = \frac{a}{z^{a-p}} \int_0^z t^{a-p-1} f(t)\mathrm{d}t \tag{2.9}$$

is in the class $\mathcal{P}_{a+1,c}(A, B; p, \lambda)$.

Proof. Making use of (2.9), we have

$$af(z) = (a-p)F(z) + zF'(z), \tag{2.10}$$

which, in the light of (1.6), yields

$$\begin{aligned} aL_p(a,c)f(z) &= (a-p)L_p(a,c)F(z) + z(L_p(a,c)F(z))' \\ &= aL_p(a+1,c)F(z). \end{aligned}$$

Therefore, we have

$$L_p(a,c)f(z) = L_p(a+1,c)F(z),$$

and the desired result follows at once. □

3. Basic properties of the function class $\mathcal{P}^+_{a,c}(A, B; p, \lambda)$

We first determine a necessary and sufficient condition for a function $f(z) \in \mathcal{A}(p)$ of the form (1.10) to be in the class $\mathcal{P}^+_{a,c}(A, B; p, \lambda)$.

Theorem 4. *Let the function $f(z) \in \mathcal{A}(p)$ be given by (1.10). Then $f(z) \in \mathcal{P}^+_{a,c}(A, B; p, \lambda)$ if and only if*

$$\sum_{k=1}^{\infty} (k+p)(1+B)\frac{(a)_k}{(c)_k}|a_{k+p}| \leqq (B-A)(p-\lambda). \tag{3.1}$$

Proof. If the condition (3.1) holds true, we find from (1.10) and (3.1) that

$$\begin{aligned} &|(L_p(a,c)f(z))' - pz^{p-1}| - |B(L_p(a,c)f(z))' - z^{p-1}[pB + (A-B)(p-\lambda)]| \\ &\quad = \left| -\sum_{k=1}^{\infty}(k+p)\frac{(a)_k}{(c)_k}|a_{k+p}|z^{k+p-1} \right| - \left| (B-A)(p-\lambda)z^{p-1} - B\sum_{k=1}^{\infty}(k+p)\frac{(a)_k}{(c)_k}|a_{k+p}|z^{k+p-1} \right| \\ &\quad \leqq \sum_{k=1}^{\infty}(k+p)(1+B)\frac{(a)_k}{(c)_k}|a_{k+p}| - (B-A)(p-\lambda) \leqq 0 \quad (z \in \partial\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| = 1\}). \end{aligned}$$

Hence, by the *Maximum Modulus Theorem*, we have

$$f(z) \in \mathcal{P}^+_{a,c}(A, B; p, \lambda).$$

Conversely, let $f(z) \in \mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$ be given by (1.10). Then, from (1.9) and (1.10), we find that

$$\left| \frac{\frac{(L_p(a,c)f(z))'}{z^{p-1}} - p}{B\,\frac{(L_p(a,c)f(z))'}{z^{p-1}} - [pB + (A-B)(p-\lambda)]} \right|$$

$$= \left| \frac{\sum_{k=1}^{\infty}(k+p)\frac{(a)_k}{(c)_k}|a_{k+p}|z^{k+p-1}}{(B-A)(p-\lambda)z^{p-1} - B\sum_{k=1}^{\infty}(k+p)\frac{(a)_k}{(c)_k}|a_{k+p}|z^{k+p-1}} \right| < 1 \quad (z \in \mathbb{U}). \tag{3.2}$$

Now, since $|\Re(z)| \leqq |z|$ for all z, we have

$$\Re\left(\frac{\sum_{k=1}^{\infty}(k+p)\frac{(a)_k}{(c)_k}|a_{k+p}|z^{k+p-1}}{(B-A)(p-\lambda)z^{p-1} - B\sum_{k=1}^{\infty}(k+p)\frac{(a)_k}{(c)_k}|a_{k+p}|z^{k+p-1}} \right) < 1. \tag{3.3}$$

We choose values of z on the real axis so that the following expression:

$$\frac{(L_p(a,c)f(z))'}{z^{p-1}}$$

is real. Then, upon clearing the denominator in (3.3) and letting $z \to 1-$ through *real* values, we get

$$\sum_{k=1}^{\infty}(k+p)(1+B)\,\frac{(a)_k}{(c)_k}\,|a_{k+p}| \leqq (B-A)(p-\lambda).$$

This completes the proof of Theorem 4. $\square$

Remark 2. Since $\mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$ is contained in the function class $\mathcal{P}_{a,c}(A, B; p, \lambda)$, a sufficient condition for $f(z)$ defined by (1.1) to be in the class $\mathcal{P}_{a,c}(A, B; p, \lambda)$ is that it satisfies the condition (3.1) of Theorem 4.

Corollary 1. *Let the function $f(z) \in \mathcal{A}(p)$ be given by (1.10). If $f(z) \in \mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$, then*

$$|a_{k+p}| \leqq \frac{(B-A)(p-\lambda)(c)_k}{(k+p)(1+B)(a)_k} \quad (k, p \in \mathbb{N}). \tag{3.4}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{(B-A)(p-\lambda)(c)_k}{(k+p)(1+B)(a)_k}z^{k+p} \quad (k, p \in \mathbb{N}). \tag{3.5}$$

We next prove the following growth and distortion properties for the class $\mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$.

Theorem 5. *If a function $f(z)$ defined by (1.10) is in the class $\mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$, then*

$$\begin{aligned} \left(\frac{p!}{(p-m)!} - \frac{c(B-A)(p-\lambda)p!}{a(1+B)(p+1-m)!}|z|\right)|z|^{p-m} &\leqq |f^{(m)}(z)| \\ &\leqq \left(\frac{p!}{(p-m)!} + \frac{c(B-A)(p-\lambda)p!}{a(1+B)(p+1-m)!}|z|\right)|z|^{p-m} \\ &(z \in \mathbb{U};\ a \geqq c > 0;\ m \in \mathbb{N}_0;\ p > m). \end{aligned} \tag{3.6}$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^p - \frac{c(B-A)(p-\lambda)}{a(1+p)(1+B)}\,z^{p+1} \quad (p \in \mathbb{N}). \tag{3.7}$$

Proof. In view of Theorem 4, we have

$$\frac{a(p+1)(1+B)}{c(B-A)(p-\lambda)(p+1)!}\sum_{k=1}^{\infty}(k+p)!|a_{k+p}| \leqq \sum_{k=1}^{\infty}\frac{(k+p)(1+B)}{(B-A)(p-\lambda)}\frac{(a)_k}{(c)_k}|a_{k+p}| \leqq 1,$$

which readily yields

$$\sum_{k=1}^{\infty}(k+p)!|a_{k+p}| \leqq \frac{c(B-A)(p-\lambda)p!}{a(1+B)} \quad (k, p \in \mathbb{N}). \tag{3.8}$$

Now, by differentiating both sides of (1.10) m times with respect to z, we obtain

$$f^{(m)}(z) = \frac{p!}{(p-m)!}z^{p-m} - \sum_{k=1}^{\infty}\frac{(k+p)!}{(k+p-m)!}|a_{k+p}|z^{k+p-m} \quad (k, p \in \mathbb{N}; m \in \mathbb{N}_0; p > m). \tag{3.9}$$

Theorem 5 follows readily from (3.8) and (3.9).

Finally, it is easy to see that the bounds in (3.6) are attained for the function $f(z)$ given by (3.7). □

4. Properties involving neighborhoods

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [25] and Ruscheweyh [26] and (more recently) by Altintaş et al. [27–29] and Aouf [30], we begin by introducing here the δ-neighborhood of a function $f(z) \in \mathcal{A}(p)$ of the form (1.1) by means of Definition 1 below.

Definition 1. For $\delta > 0$, $a > 0$, $c > 0$ and a non-negative sequence $T = \{t_k\}_{k=1}^{\infty}$, where

$$t_k := \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} \quad (k \in \mathbb{N}),$$

the δ-neighborhood of a function $f(z) \in \mathcal{A}(p)$ of the form (1.1) is defined as follows:

$$N_\delta(f) := \left\{g : g(z) = z^p + \sum_{k=1}^{\infty}b_{k+p}z^{k+p} \in \mathcal{A}(p) \text{ and } \sum_{k=1}^{\infty}t_k|b_{k+p} - a_{k+p}| \leqq \delta \ (\delta > 0; a > 0; c > 0)\right\}. \tag{4.1}$$

We now prove our first result based upon the familiar concept of neighborhood defined by (4.1).

Theorem 6. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{P}_{a,c}(A, B; p, \lambda)$. If $f(z)$ satisfies the following condition:*

$$\frac{f(z) + \varepsilon z^p}{1+\varepsilon} \in \mathcal{P}_{a,c}(A, B; p, \lambda) \quad (\varepsilon \in \mathbb{C}; |\varepsilon| < \delta; \delta > 0), \tag{4.2}$$

then

$$N_\delta(f) \subset \mathcal{P}_{a,c}(A, B; p, \lambda). \tag{4.3}$$

Proof. It is easily seen from (1.9) that $g(z) \in \mathcal{P}_{a,c}(A, B; p, \lambda)$ if and only if, for any complex σ ($|\sigma| = 1$),

$$\frac{(L_p(a,c)g(z))' - pz^{p-1}}{B(L_p(a,c)g(z))' - z^{p-1}[pB + (A-B)(p-\lambda)]} \neq \sigma \quad (z \in \mathbb{U}; \sigma \in \mathbb{C}; |\sigma| = 1), \tag{4.4}$$

which is equivalent to the following inequality:

$$\frac{(g * h)(z)}{z^p} \neq 0 \quad (z \in \mathbb{U}), \tag{4.5}$$

where, for convenience,

$$h(z) = z^p + \sum_{k=1}^{\infty} c_{k+p} z^{k+p} = z^p + \sum_{k=1}^{\infty} \frac{(k+p)(1+\sigma B)(a)_k}{\sigma(A-B)(p-\lambda)(c)_k} z^{k+p}. \tag{4.6}$$

It follows from (4.6) that

$$|c_{k+p}| \leqq \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} \quad (k \in \mathbb{N}). \tag{4.7}$$

Now, if $f(z) \in \mathcal{A}(p)$, given by (1.1), satisfies the condition (4.2), then (4.5) yields

$$\frac{\left(\frac{f(z)+\varepsilon z^p}{1+\varepsilon}\right) * h(z)}{z^p} \neq 0 \quad (z \in \mathbb{U})$$

or

$$\frac{f(z) * h(z)}{z^p} \neq -\varepsilon \quad (z \in \mathbb{U}),$$

which is equivalent to the following inequality:

$$\left|\frac{f(z) * h(z)}{z^p}\right| \geqq \delta \quad (z \in \mathbb{U};\ \delta > 0). \tag{4.8}$$

By letting

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p} \in N_\delta(f), \tag{4.9}$$

we deduce that

$$\left|\frac{[f(z) - g(z)] * h(z)}{z^p}\right| = \left|\sum_{k=1}^{\infty} (b_{k+p} - a_{k+p}) c_{k+p} z^k\right| \leqq \sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} |b_{k+p} - a_{k+p}| < \delta \quad (z \in \mathbb{U}; \delta > 0), \tag{4.10}$$

which leads us to (4.5), and hence also to (4.4) for any $\sigma \in \mathbb{C}$ $(|\sigma| = 1)$. This implies that $g(z) \in \mathcal{P}_{a,c}(A, B; p, \lambda)$, which completes the proof of the assertion (4.3) of Theorem 6. □

We now define the δ-neighborhood of a function $f(z) \in \mathcal{A}(p)$ of the form (1.10) as follows.

Definition 2. For $\min\{\delta, a, c\} > 0$, the δ-neighborhood of a function $f(z) \in \mathcal{A}(p)$ of the form (1.10) is given by

$$N_\delta^+(f) := \left\{ g : g(z) = z^p - \sum_{k=1}^{\infty} |b_{k+p}| z^{k+p} \in \mathcal{A}(p) \text{ and } \sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} \big||b_{k+p}| - |a_{k+p}|\big| \leqq \delta \ (\delta > 0; a > 0; c > 0) \right\}. \tag{4.11}$$

Theorem 7. *If the function $f(z)$ defined by (1.10) is in the class $\mathcal{P}_{a+1,c}^+(A, B; p, \lambda)$, then*

$$N_\delta^+(f) \subset \mathcal{P}_{a,c}^+(A, B; p, \lambda) \quad \left(\delta := \frac{1}{a+1}\right). \tag{4.12}$$

The result is the best possible in the sense that δ cannot be increased.

Proof. Let $f(z) \in \mathcal{P}^{+}_{a+1,c}(A, B; p, \lambda)$ be given by (1.10). Then, by Theorem 4, we have

$$\sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} |a_{k+p}| \leqq \frac{a}{a+1}. \tag{4.13}$$

Similarly, by taking

$$g(z) = z^p - \sum_{k=1}^{\infty} |b_{k+p}| z^{k+p} \in N^{+}_{\delta}(f) \quad \left(\delta = \frac{1}{a+1}\right), \tag{4.14}$$

we find from the definition (4.11) that

$$\sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} ||b_{k+p}| - |a_{k+p}|| \leqq \delta \quad (\delta > 0). \tag{4.15}$$

With the help of (4.13) and (4.15), we get

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} |b_{k+p}| &\leqq \sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} |a_{k+p}| \\ &\quad + \sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} ||b_{k+p}| - |a_{k+p}|| \\ &\leqq \frac{a}{a+1} + \delta = 1. \end{aligned}$$

Thus, in view of Theorem 4 again, we see that $g(z) \in \mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$.

To show the sharpness of the assertion of Theorem 7, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z) = z^p - \frac{c(B-A)(p-\lambda)}{(a+1)(1+p)(1+B)} z^{p+1} \in \mathcal{P}^{+}_{a+1,c}(A, B; p, \lambda) \tag{4.16}$$

and

$$g(z) = z^p - \left(\frac{c(B-A)(p-\lambda)}{(a+1)(1+p)(1+B)} + \frac{c(B-A)(p-\lambda)\delta'}{a(1+p)(1+B)} \right) z^{p+1}, \tag{4.17}$$

where

$$\delta' > \delta = \frac{1}{a+1}.$$

Clearly, the function $g(z)$ belongs to $N^{+}_{\delta'}(f)$. On the other hand, we find from Theorem 4 that $g(z) \notin \mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$. The proof of Theorem 7 is thus completed. □

5. Properties associated with modified Hadamard products

For the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z^p - \sum_{k=1}^{\infty} |a_{k+p,j}| z^{k+p} \quad (j = 1, 2;\ p \in \mathbb{N}), \tag{5.1}$$

we denote by $(f_1 \bullet f_2)(z)$ the *modified* Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, defined by

$$(f_1 \bullet f_2)(z) := z^p - \sum_{k=1}^{\infty} |a_{k+p,1}| \cdot |a_{k+p,2}| z^{k+p} =: (f_2 \bullet f_1)(z). \tag{5.2}$$

Theorem 8. *Let the functions* $f_j(z)$ $(j = 1, 2)$ *defined by* (5.1) *be in the class* $\mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$. *Then*

$$(f_1 \bullet f_2)(z) \in \mathcal{P}^{+}_{a,c}(A, B; p, \gamma),$$

where

$$\gamma := p - \frac{c(B-A)(p-\lambda)^2}{a(1+p)(1+B)}. \tag{5.3}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ *given by*

$$f_j(z) = z^p - \frac{c(B-A)(p-\lambda)}{a(1+p)(1+B)} z^{p+1} \quad (j = 1, 2;\ p \in \mathbb{N}). \tag{5.4}$$

Proof. Employing the technique used earlier by Schild and Silverman [31], we need to find the largest γ such that

$$\sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\gamma)(c)_k} |a_{k+p,1}| \cdot |a_{k+p,2}| \leqq 1 \quad (f_j(z) \in \mathcal{P}^{+}_{a,c}(A, B; p, \lambda)(j = 1, 2)). \tag{5.5}$$

Since $f_j(z) \in \mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$ $(j = 1, 2)$, we readily see that

$$\sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} |a_{k+p,j}| \leqq 1 \quad (j = 1, 2). \tag{5.6}$$

Therefore, by the Cauchy–Schwarz inequality, we obtain

$$\sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} \sqrt{|a_{k+p,1}| \cdot |a_{k+p,2}|} \leqq 1. \tag{5.7}$$

This implies that we only need to show that

$$\frac{|a_{k+p,1}| \cdot |a_{k+p,2}|}{p-\gamma} \leqq \frac{\sqrt{|a_{k+p,1}| \cdot |a_{k+p,2}|}}{p-\lambda} \quad (k \in \mathbb{N}) \tag{5.8}$$

or, equivalently, that

$$\sqrt{|a_{k+p,1}| \cdot |a_{k+p,2}|} \leqq \frac{p-\gamma}{p-\lambda} \quad (k \in \mathbb{N}). \tag{5.9}$$

Hence, by making use of the inequality (5.7), it is sufficient to prove that

$$\frac{(B-A)(p-\lambda)(c)_k}{(k+p)(1+B)(a)_k} \leqq \frac{p-\gamma}{p-\lambda} \quad (k \in \mathbb{N}), \tag{5.10}$$

that is, that

$$\gamma \leqq p - \frac{(B-A)(p-\lambda)^2(c)_k}{(k+p)(1+B)(a)_k} \quad (k \in \mathbb{N}). \tag{5.11}$$

Now, defining the function $\Phi(k)$ by

$$\Phi(k) := p - \frac{(B-A)(p-\lambda)^2(c)_k}{(k+p)(1+B)(a)_k} \quad (k \in \mathbb{N}), \tag{5.12}$$

we see that $\Phi(k)$ is an *increasing* function of k. Therefore, we conclude that

$$\gamma \leqq \Phi(1) = p - \frac{c(B-A)(p-\lambda)^2}{a(1+p)(1+B)}, \tag{5.13}$$

which completes the proof of Theorem 8. □

By using arguments similar to those in the proof of Theorem 8, we can derive the following result.

Theorem 9. *Let the function $f_1(z)$ defined by (5.1) be in the class $\mathcal{P}^+_{a,c}(A, B; p, \lambda)$. Suppose also that the function $f_2(z)$ defined by (5.1) is in the class $\mathcal{P}^+_{a,c}(A, B; p, \gamma)$. Then*

$$(f_1 \bullet f_2)(z) \in \mathcal{P}^+_{a,c}(A, B; p, \xi),$$

where

$$\xi := p - \frac{c(B-A)(p-\lambda)(p-\gamma)}{a(1+p)(1+B)}. \tag{5.14}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_1(z) = z^p - \frac{c(B-A)(p-\lambda)}{a(1+p)(1+B)} z^{p+1} \quad (p \in \mathbb{N}) \tag{5.15}$$

and

$$f_2(z) = z^p - \frac{c(B-A)(p-\gamma)}{a(1+p)(1+B)} z^{p+1} \quad (p \in \mathbb{N}). \tag{5.16}$$

Theorem 10. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $\mathcal{P}^+_{a,c}(A, B; p, \lambda)$. Then the function $h(z)$ defined by*

$$h(z) = z^p - \sum_{k=1}^{\infty} (|a_{k+p,1}|^2 + |a_{k+p,2}|^2) z^{k+p} \tag{5.17}$$

belongs to the class $\mathcal{P}^+_{a,c}(A, B; p, \chi)$, where

$$\chi := p - \frac{2c(B-A)(p-\lambda)^2}{a(1+p)(1+B)}. \tag{5.18}$$

This result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by (5.4).

Proof. By noting that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(k+p)^2(1+B)^2}{(B-A)^2(p-\lambda)^2} \left(\frac{(a)_k}{(c)_k}\right)^2 |a_{k+p,j}|^2 &\leqq \left(\sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k} |a_{k+p,j}|\right)^2 \\ &\leqq 1 \quad (j = 1, 2) \\ &\quad (f_j(z) \in P^+_{a,c}(A, B; p, \lambda)\ (j = 1, 2)), \end{aligned} \tag{5.19}$$

we have

$$\sum_{k=1}^{\infty} \frac{(k+p)^2(1+B)^2}{(B-A)^2(p-\lambda)^2} \left(\frac{(a)_k}{(c)_k}\right)^2 \left(|a_{k+p,1}|^2 + |a_{k+p,2}|^2\right) \leqq 1. \tag{5.20}$$

Therefore, we have to find the largest χ such that

$$\frac{1}{p-\chi} \leqq \frac{(k+p)(1+B)(a)_k}{2(B-A)(p-\lambda)(c)_k} \quad (k \in \mathbb{N}), \tag{5.21}$$

that is, that

$$\chi \leqq p - \frac{2(B-A)(p-\lambda)^2(c)_k}{(k+p)(1+B)(a)_k} \quad (k \in \mathbb{N}). \tag{5.22}$$

Now, if we define a function $\Psi(k)$ by

$$\Psi(k) := p - \frac{2(B-A)(p-\lambda)^2(c)_k}{(k+p)(1+B)(a)_k} \quad (k \in \mathbb{N}), \tag{5.23}$$

we observe that $\Psi(k)$ is an *increasing* function of k. We thus conclude that

$$\chi \leqq \Psi(1) = p - \frac{2c(B-A)(p-\lambda)^2}{a(1+p)(1+B)}, \tag{5.24}$$

which completes the proof of Theorem 10. □

6. Applications of fractional calculus operators

Various operators of fractional calculus (that is, fractional derivatives and fractional integrals) have been studied in the literature rather extensively (*cf.*, *e.g.*, [22,23]; see also [32,33]). In our investigation, we propose to apply the fractional calculus operators which are defined below.

Definition 3. The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} \mathrm{d}\zeta \quad (\mu > 0), \tag{6.1}$$

where the function $f(z)$ is analytic in a simply-connected domain of the complex z-plane containing the origin and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 4. The fractional derivative of order μ is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} \mathrm{d}\zeta \quad (0 \leqq \mu < 1), \tag{6.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 3.

Definition 5. Under the hypotheses of Definition 4, the fractional derivative of order $n+\mu$ is defined, for a function $f(z)$, by

$$D_z^{n+\mu} f(z) = \frac{\mathrm{d}^n}{\mathrm{d}z^n}\{D_z^{\mu} f(z)\} \quad (0 \leqq \mu < 1;\ n \in \mathbb{N}_0). \tag{6.3}$$

In this section we shall investigate the growth and distortion properties of functions in the class $\mathcal{P}_{a,c}^{+}(A, B; p, \lambda)$, which involve the operators $J_{\delta,p}$ and D_z^{μ}. We shall need the following lemma given by Chen et al. [32].

Lemma 2 (*Chen et al. [32]*). *For a function* $f(z) \in \mathcal{A}(p)$,

$$D_z^{\mu}\{(J_{\delta,p} f)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} z^{p-\mu} - \sum_{k=1}^{\infty} \frac{(\delta+p)\Gamma(k+p+1)}{(\delta+k+p)\Gamma(k+p-\mu+1)} a_{k+p} z^{k+p-\mu} \quad (\mu \in \mathbb{R};\ \delta > -p;\ p \in \mathbb{N}) \tag{6.4}$$

and

$$J_{\delta,p}(D_z^{\mu}\{f(z)\}) = \frac{(\delta+p)\Gamma(p+1)}{(\delta+p-\mu)\Gamma(p-\mu+1)} z^{p-\mu} - \sum_{k=1}^{\infty} \frac{(\delta+p)\Gamma(k+p+1)}{(\delta+k-\mu)\Gamma(k+p-\mu+1)} a_{k+p} z^{k+p-\mu} \quad (\mu \in \mathbb{R};\ \delta > -p;\ p \in \mathbb{N}) \tag{6.5}$$

provided that no zeros appear in the denominators in (6.4) and (6.5).

Remark 3. Throughout this section, we assume *further* that

$$a \geqq c > 0.$$

Theorem 11. *Let the function $f(z)$ defined by (1.10) be in the class $\mathcal{P}^{+}_{a,c}(A, B; p, \lambda)$. Then*

$$|D_z^{-\mu}\{(J_{\delta,p} f)(z)\}| \geqq \frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}|z|^{p+\mu} \cdot \left(1 - \frac{c(\delta+p)(B-A)(p-\lambda)}{a(\delta+1+p)(p+1+\mu)(1+B)}|z|\right)$$
$$(\mu > 0;\ \delta > -p;\ p \in \mathbb{N};\ z \in \mathbb{U}) \tag{6.6}$$

and

$$|D_z^{-\mu}\{(J_{\delta,p} f)(z)\}| \leqq \frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}|z|^{p+\mu} \cdot \left(1 + \frac{c(\delta+p)(B-A)(p-\lambda)}{a(\delta+1+p)(p+1+\mu)(1+B)}|z|\right)$$
$$(\mu > 0;\ \delta > -p;\ p \in \mathbb{N};\ z \in \mathbb{U}). \tag{6.7}$$

Each of the assertions (6.6) and (6.7) is sharp.

Proof. In view of Theorem 4, we have

$$\frac{a(1+p)(1+B)}{c(B-A)(p-\lambda)} \sum_{k=1}^{\infty} |a_{k+p}| \leqq \sum_{k=1}^{\infty} \frac{(k+p)(1+B)(a)_k}{(B-A)(p-\lambda)(c)_k}|a_{k+p}| \leqq 1, \tag{6.8}$$

which readily yields

$$\sum_{k=1}^{\infty} |a_{k+p}| \leqq \frac{c(B-A)(p-\lambda)}{a(1+p)(1+B)}. \tag{6.9}$$

Consider the function $G(z)$ defined in $\mathbb{U}$ by

$$\begin{aligned} G(z) &:= \frac{\Gamma(p+1+\mu)}{\Gamma(p+1)}\, z^{-\mu} D_z^{-\mu}\left\{(J_{\delta,p} f)(z)\right\} \\ &= z^p - \sum_{k=1}^{\infty} \frac{(\delta+p)\Gamma(k+p+1)\Gamma(p+1+\mu)}{(\delta+k+p)\Gamma(k+p+1+\mu)\Gamma(p+1)}|a_{k+p}|z^{k+p} \\ &= z^p - \sum_{k=1}^{\infty} \Theta(k)|a_{k+p}|z^{k+p} \quad (z \in \mathbb{U}), \end{aligned}$$

where

$$\Theta(k) := \frac{(\delta+p)\Gamma(k+p+1)\Gamma(p+1+\mu)}{(\delta+k+p)\Gamma(k+p+1+\mu)\Gamma(p+1)} \quad (k, p \in \mathbb{N};\ \mu > 0). \tag{6.10}$$

Since $\Theta(k)$ is a *decreasing* function of k when $\mu > 0$, we get

$$0 < \Theta(k) \leqq \Theta(1) = \frac{(\delta+p)(p+1)}{(\delta+p+1)(p+1+\mu)} \quad (\delta > -p;\ p \in \mathbb{N};\ \mu > 0). \tag{6.11}$$

Thus, by using (6.9) and (6.11), we deduce that

$$\begin{aligned} |G(z)| &\geqq |z|^p - \Theta(1)\,|z|^{p+1} \sum_{k=1}^{\infty} |a_{k+p}| \\ &\geqq |z|^p - \frac{c(\delta+p)(B-A)(p-\lambda)}{a(\delta+1+p)(p+1+\mu)(1+B)}|z|^{p+1} \quad (z \in \mathbb{U}) \end{aligned}$$

and

$$\begin{aligned} |G(z)| &\leqq |z|^p + \Theta(1)\,|z|^{p+1} \sum_{k=1}^{\infty} |a_{k+p}| \\ &\leqq |z|^p + \frac{c(\delta+p)(B-A)(p-\lambda)}{a(\delta+1+p)(p+1+\mu)(1+B)}|z|^{p+1} \quad (z \in \mathbb{U}), \end{aligned}$$

which yield the inequalities (6.6) and (6.7) of Theorem 11. Equalities in (6.6) and (6.7) are attained for the function $f(z)$ given by

$$D_z^{-\mu}\{(J_{\delta,p}f)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p+1+\mu)}\, z^{p+\mu} \cdot \left(1 - \frac{c(\delta+p)(B-A)(p-\lambda)}{a(\delta+1+p)(p+1+\mu)(1+B)} z\right) \tag{6.12}$$

or, equivalently, by

$$(J_{\delta,p}f)(z) = z^p - \frac{c(\delta+p)(B-A)(p-\lambda)}{a(\delta+1+p)(p+1)(1+B)} z^{p+1}. \tag{6.13}$$

We thus have completed the proof of Theorem 11. □

Theorem 12. *Let the function $f(z)$ defined by (1.10) be in the class $\mathcal{P}^+_{a,c}(A, B; p, \lambda)$. Then*

$$|D_z^{\mu}\{(J_{\delta,p}f)(z)\}| \geqq \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)} |z|^{p-\mu} \cdot \left(1 - \frac{c(\delta+p)(B-A)(p-\lambda)}{a(\delta+1+p)(p+1-\mu)(1+B)} |z|\right)$$
$$(0 \leqq \mu < 1;\ \delta > -p;\ p \in \mathbb{N};\ z \in \mathbb{U}) \tag{6.14}$$

and

$$|D_z^{\mu}\{(J_{\delta,p}f)(z)\}| \leqq \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}\, |z|^{p-\mu} \cdot \left(1 + \frac{c(\delta+p)(B-A)(p-\lambda)}{a(\delta+1+p)(p+1-\mu)(1+B)}\, |z|\right)$$
$$(0 \leqq \mu < 1;\ \delta > -p;\ p \in \mathbb{N};\ z \in \mathbb{U}). \tag{6.15}$$

Each of the assertions (6.14) and (6.15) is sharp.

Proof. Since $a \geqq c > 0$, it follows from Theorem 4 that

$$\sum_{k=1}^{\infty}(k+p)|a_{k+p}| \leqq \frac{c(B-A)(p-\lambda)}{a(1+B)}. \tag{6.16}$$

We now consider the function $H(z)$ defined in $\mathbb{U}$ by

$$\begin{aligned} H(z) &= \frac{\Gamma(p+1-\mu)}{\Gamma(p+1)}\, z^{\mu} D_z^{\mu}\left\{(J_{\delta,p}f)(z)\right\} \\ &= z^p - \sum_{k=1}^{\infty} \frac{(\delta+p)\Gamma(k+p)\Gamma(p+1-\mu)}{(\delta+k+p)\Gamma(k+p+1-\mu)\Gamma(p+1)}(k+p)|a_{k+p}|z^{k+p} \\ &= z^p - \sum_{k=1}^{\infty} \Xi(k)(k+p)|a_{k+p}|z^{k+p} \quad (z \in \mathbb{U}), \end{aligned}$$

where

$$\Xi(k) := \frac{(\delta+p)\Gamma(k+p)\Gamma(p+1-\mu)}{(\delta+k+p)\Gamma(k+p+1-\mu)\Gamma(p+1)} \quad (k, p \in \mathbb{N};\ 0 \leqq \mu < 1).$$

Since $\Xi(k)$ is a *decreasing* function of k when $0 \leqq \mu < 1$, we find that

$$0 < \Xi(k) \leqq \Xi(1) = \frac{\delta+p}{(\delta+1+p)(p+1-\mu)} \quad (\delta > -p;\ p \in \mathbb{N};\ 0 \leqq \mu < 1). \tag{6.17}$$

Consequently, with the aid of (6.16) and (6.17), we obtain

$$\begin{aligned} |H(z)| &\geqq |z|^p - \Xi(1)\,|z|^p \sum_{k=1}^{\infty}(k+p)|a_{k+p}| \\ &\geqq |z|^p - \frac{c(\delta+p)}{a(\delta+1+p)(p+1-\mu)(1+B)}\, |z|^{p+1} \quad (z \in \mathbb{U}) \end{aligned}$$

and

$$
\begin{aligned}
|H(z)| &\leqq |z|^p + \Xi(1)|z|^{p+1}\sum_{k=1}^{\infty}(k+p)|a_{k+p}| \\
&\leqq |z|^p + \frac{c(\delta+p)}{a(\delta+1+p)(p+1-\mu)(1+B)}|z|^{p+1} \quad (z \in \mathbb{U}),
\end{aligned}
$$

which yield the inequalities (6.14) and (6.15) of Theorem 12.

Finally, since the equalities in (6.14) and (6.15) are attained for the function $f(z)$ given by

$$
D_z^{\mu}\{(J_{\delta,p}f)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p+1-\mu)}z^{p-\mu}\cdot\left(1-\frac{c(\delta+p)(B-A)(p-\lambda)}{a(\delta+1+p)(p+1-\mu)(1+B)}z\right) \tag{6.18}
$$

or, equivalently, for the function $(J_{\delta,p}f)(z)$ given by (6.13), the proof of Theorem 12 is thus completed. □

Acknowledgements

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

References

[1] H. Saitoh, A linear operator and its applications of first order differential subordinations, Math. Japon. 44 (1996) 31–38.
[2] H.M. Srivastava, J. Patel, Some subclasses of multivalent functions involving a certain linear operator, J. Math. Anal. Appl. 310 (2005) 209–228.
[3] V. Kumar, S.L. Shukla, Multivalent functions defined by Ruscheweyh derivatives, Indian J. Pure Appl. Math. 15 (1984) 1216–1227.
[4] V. Kumar, S.L. Shukla, Multivalent functions defined by Ruscheweyh derivatives. II, Indian J. Pure Appl. Math. 15 (1984) 1228–1238.
[5] M.K. Aouf, On certain subclass of analytic p-valent functions. II, Math. Japon. 34 (1989) 683–691.
[6] S. Owa, On certain subclass of analytic p-valent functions, Math. Japon. 29 (1984) 191–198.
[7] M.-P. Chen, A class of p-valent functions, Soochow J. Math. 8 (1982) 15–26.
[8] R.M. Goel, N.S. Sohi, New criteria for p-valence, Indian J. Pure Appl. Math. 11 (1980) 1356–1360.
[9] M.K. Aouf, A generalization of multivalent functions defined by Ruscheweyh derivatives, Soochow J. Math. 17 (1991) 83–97.
[10] B.S. Mehrok, A class of univalent functions, Tamkang J. Math. 13 (1982) 141–155.
[11] M.K. Aouf, A generalization of multivalent functions with negative coefficients. II, Bull. Korean Math. Soc. 25 (1988) 221–232.
[12] M.K. Aouf, Certain classes of p-valent functions with negative coefficients. II, Indian J. Pure Appl. Math. 19 (1988) 761–767.
[13] S.L. Shukla, Dashrath, On certain classes of multivalent functions with negative coefficients, Soochow J. Math. 8 (1982) 179–188.
[14] S.K. Lee, S. Owa, H.M. Srivastava, Basic properties and characterizations of a certain class of analytic functions with negative coefficients, Utilitas Math. 36 (1989) 121–128.
[15] V.P. Gupta, P.K. Jain, Certain classes of univalent functions with negative coefficients. II, Bull. Austral. Math. Soc. 14 (1976) 467–473.
[16] M.K. Aouf, H.E. Darwish, Some classes of multivalent functions with negative coefficients. I, Honam Math. J. 16 (1994) 119–135.
[17] B.A. Uralegaddi, S.M. Sarangi, Some classes of univalent functions with negative coefficients, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Seçt. I a Mat. (N. S.) 34 (1988) 7–11.
[18] S.D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc. 135 (1969) 429–446.
[19] R.J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc. 16 (1969) 755–758.
[20] A.E. Livingston, On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 17 (1966) 352–357.
[21] H.M. Srivastava, S. Owa (Eds.), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, 1992.
[22] S. Owa, On distortion theorems. I, Kyungpook Math. J. 18 (1978) 55–59.
[23] H.M. Srivastava, S. Owa (Eds.), Univalent Functions, Fractional Calculus, and their Applications, Halsted Press (Ellis Horwood Limited, Chichester), 1989, John Wiley and Sons, New York.
[24] I.S. Jack, Functions starlike and convex functions of order α, J. London Math. Soc. (2) 2 (1971) 469–474.
[25] A.W. Goodman, Univalent functions and nonanalytic curves, Proc. Amer. Math. Soc. 8 (1957) 598–601.
[26] S. Ruscheweyh, Neighborhoods of univalent functions, Proc. Amer. Math. Soc. 81 (1981) 521–527.
[27] O. Altintaş, S. Owa, Neighborhoods of certain analytic functions with negative coefficients, Internat. J. Math. Math. Sci. 19 (1996) 797–800.
[28] O. Altintaş, Ö. Özkan, H.M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, Appl. Math. Lett. 13 (3) (2000) 63–67.
[29] O. Altintaş, Ö. Özkan, H.M. Srivastava, Neighborhoods of a certain family of multivalent functions with negative coefficients, Comput. Math. Appl. 47 (2004) 1667–1672.
[30] M.K. Aouf, Neighborhoods of certain classes of analytic functions with negative coefficients, Internat. J. Math. Math. Sci. 2006 (2006) 1–6.
[31] A. Schild, H. Silverman, Convolution of univalent functions with negative coefficients, Ann. Univ. Mariae Curie-Skłodowska Sect. A 29 (1975) 99–107.
[32] M.-P. Chen, H. Irmark, H.M. Srivastava, Some families of multivalently analytic functions with negative coefficients, J. Math. Anal. Appl. 214 (1997) 674–690.
[33] H.M. Srivastava, M.K. Aouf, A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. I and II, J. Math. Anal. Appl. 171 (1992) 1–13; J. Math. Anal. Appl. 192 (1995) 673–688.